JOURNAL OF COMPUTATIONAL PHYSICS 8, 304–308 (1971)

Note

Root Finding of Polynomials as an Initial Value Problem

D. H. Lehmer sums up the present state of the art of solving polynomials as follows [1]:

> "The problem of solving polynomial equations is not only quite old but is also slow to modify itself to the age of computing. The more popular methods in use by modern computers are simply adaptations of the classical hand methods of the last two centuries. Once we allow ourselves the generality of equations with complex coefficients there is not any set of n complex numbers that cannot be the "solution set" of some polynomial. Hence there is no reason why equation solving cannot be regarded as a search problem in two dimensions. In this light it is obvious that for a digital computer with a finite precision and a limited amount of time, the problem of locating n arbitrarily situated points is indeed difficult."

In this note we shall show how to find all the roots of the N-degree polynomial

$$f_1(z_1) = \sum_{n=0}^{N} a_n^{(1)} {z_1}^n = 0, \tag{1}$$

where the coefficients may be complex and $a_N^{(1)} = 1$, following a method first suggested by Davidenko [2], which is particularly suitable for computers. Let us define an imbedding

$$f(z, t) = \sum_{n=0}^{N} a_n(t)[z(t)]^n = 0, \tag{2}$$

with $a_N(t) = 1$, and with

$$a_n(0) = a_n^{(0)}, \tag{3}$$

$$a_n(1) = a_n^{(1)}, \tag{4}$$

and where the $\dot{a}_n(t)$ are continuous functions. Here and in what follows dots denote t-derivatives. The coefficients in Eq. (3) are chosen so that the roots of the polynomial

$$f_0(z_0) = \sum_{n=0}^{N} a_n^{(0)} z_0^{\,n} = 0, \tag{5}$$

which we call the "base" can somehow be found. For example, the "base"

$$z_0^N + 1 = 0 \tag{6}$$

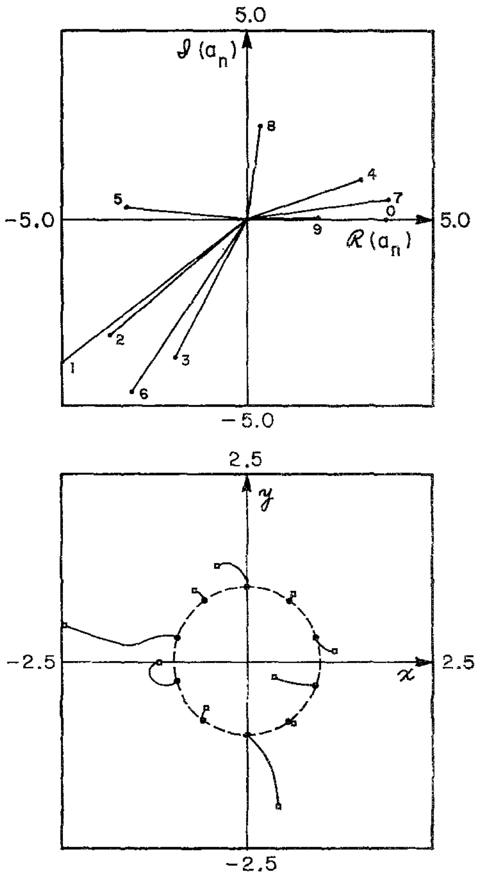

FIG. 1. Trajectories of the coefficients (upper) and roots (lower) for the 10-degree polynomial.

has the solutions

$$z_0^{(j)} = e^{i\theta_j}, \qquad \theta_j = \frac{(2j-1)\pi}{N}, \tag{7}$$

where $j = 1, 2, \ldots, N$.

Next we derive a differential equation which describes the development of the roots $z^{(j)}(t)$ in Eq. (2), $0 \leqslant t \leqslant 1$, $j = 1, 2, \ldots, N$. For example, if the coefficients vary linearly

$$a_n(t) = a_n^{(0)} - (a_n^{(0)} - a_n^{(1)})t, \qquad n = 0, 1, \ldots, N-1, \tag{8}$$

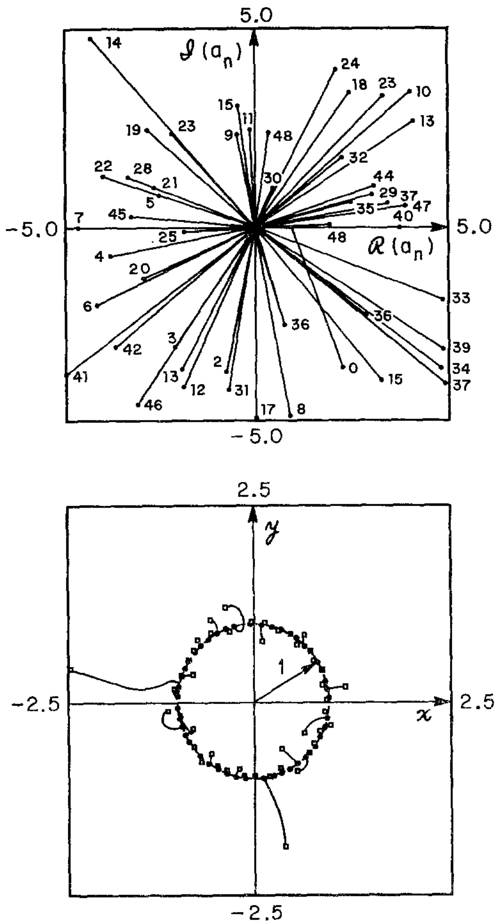

FIG. 2. Trajectories of the coefficients (upper) and roots (lower) for the 50-degree polynomial.

then the corresponding differential equation is

$$\dot{z}^{(j)} = \frac{\sum_{n=0}^{N-1} (a_n^{(0)} - a_n^{(1)})(z^{(j)})^n}{\sum_{n=1}^{N} n[a_n^{(0)} - (a_n^{(0)} - a_n^{(1)})t](z^{(j)})^{n-1}}, \qquad j = 1, 2,..., N. \tag{9}$$

Equation (9) represents a system of N uncoupled and identical differential equations, but with different initial values

$$z^{(j)}(0) = z_0^{(j)}, \qquad j = 1, 2,..., N, \tag{10}$$

where $z_0^{(j)}$ are the known roots of the base, Eq. (5).

The initial value problem defined by Eqs. (9, 10) is very suitable for solution by digital (and hybrid) computers, say by Runge–Kutta or rational extrapolation methods. Integrating from $t = 0$ until $t = 1$ we obtain

$$z^{(j)}(1) = z_1^{(j)}, \qquad j = 1, 2,..., N, \tag{11}$$

which because of Eq. (4), are the desired roots of Eq. (1). This solves our problem.

In order to demonstrate the method we computed the roots of Eq. (1), where the coefficients

$$a_n^{(1)} = a_{nr}^{(1)} + i a_{ni}^{(1)} \tag{12}$$

were distributed randomly and uniformly (by the GE subroutine URAN) in the square

$$-5 \leqslant a_{nr}^{(1)} \leqslant 5 \qquad \text{and} \qquad -5 \leqslant a_{ni}^{(1)} \leqslant 5. \tag{13}$$

We have used the base (6) and the linear variation of the coefficients given by Eq. (8). The results for $N = 10$ and $N = 50$ are displayed in Figs. 1 and 2 respectively, where we show the trajectories of the coefficients and the roots. The terminal squares correspond to the original problem.

The present method, unlike other root-finding schemes, is not iterative. However, because of truncation and round-off errors, the final values of the integration do not exactly coincide with the roots of the polynomial. In order to increase the accuracy of the results one should either increase the accuracy of the integration, or more economically, improve the final values of a rather crude integration by an efficient iterative scheme, say the Newton–Raphson method. Several important points, e.g., the case of multiple roots, imbeddings which pass through multiple roots, computation times, and others, were not treated in this short note, but can be found in an unpublished report by the author.

Acknowledgment

The author would like to thank J. McKenna and J. F. Traub for helpful discussions and comments, and Miss J. B. Seery for programming the figures.

References

1. B. Dejon and P. Henrici (Ed.) "Constructive Aspects of the Fundamental Theorem of Algebra", p. 193, Interscience, New York, 1969. (Many other references too can be found in this book.)
2. D. F. Davidenko, On a new method of numerical solution of systems of nonlinear equations (in Russian), *Dokl. Akad. Nauk. SSSR* **88** (1953), 601–602.

Received January 5, 1971

E. Watterstrom*

Bell Telephone Laboratories, Incorporated,
Murray Hill, New Jersey

* Mr. Wasserstrom has returned to the Technion — Israel Institute of Technology, Haifa, Israel.